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The effect of mean compression or dilatation on the turbulence structure of supersonic boundary layers

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It is now well known that the turbulence structure of thin shear layers can be strongly affected by the application of extra rates of strain in addition to the shear velocity gradient. Examples of such extra strain rates include lateral divergence or convergence, and streamline curvature in the plane of the mean shear. The changes in Reynolds stress are an order of magnitude larger than would be expected from the explicit extra terms in the Reynolds-stress transport equations, and therefore an order of magnitude larger than predicted by conventional calculation methods. In the present paper, one of a series on 'complex' turbulent flows, we show that bulk compression or dilatation (i.e. an extra strain rate $\operatorname{div}\mathbf{U}$) also appears to affect turbulent shear layers, typical values of Reynolds stress being increased by compression and decreased by dilatation. The fractional change in Reynolds stress is an order of magnitude larger than the fractional change in volume of a fluid element. The physical mechanism is probably analogous to that responsible for the large effects of divergence or convergence in incompressible flow. Because the phenomenon seems to be of great practical importance we discuss it in the context of engineering calculation methods. An empirical correction formula, analogous to those used to allow for divergence or curvature effects, greatly reduces the large discrepancies found between recent experiments on supersonic boundary layers and calculations by conventional extensions of successful incompressible-flow methods.

1. Introduction

Figure 1, reproduced from Bradshaw & Ferriss (1971), shows a comparison between their calculation method and the experiments of Zwarts (1970) in a boundary layer decelerating from $M_e \approx 4$ to $M_e \approx 3$. Large discrepancies appear. This calculation method is an extension of a successful method for incompressible flow (Bradshaw, Ferriss & Atwell 1967; Bradshaw & Ferriss 1972), involving Morkovin's hypothesis (Favre 1964, p. 367) that the effects of density fluctuations on the turbulence structure are small in boundary layers at non-hypersonic Mach numbers. The behaviour of constant-pressure flows is well predicted up to $M_e = 5$ at least. Most other calculation methods for compressible shear layers use Morkovin's hypothesis explicitly or implicitly, and would be expected to give results for Zwarts' flow broadly similar to those shown in figure 1. 'Eddy viscosity' methods give rather better predictions of the initial part of the flow

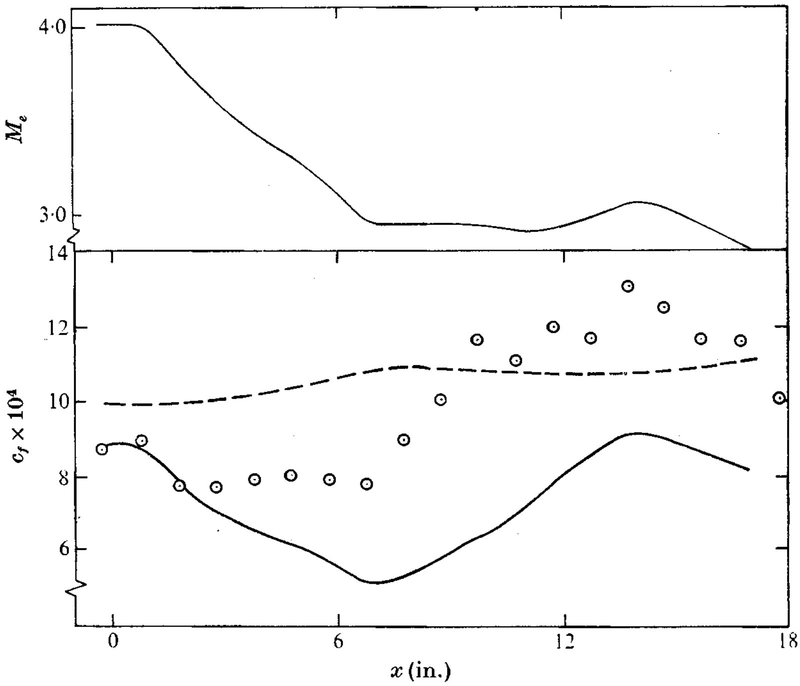

FIGURE 1. Experiment of Zwarts. ⊙, experiment; ——, calculation without allowance for dilatation; – – – –, c_f in zero pressure gradient at same Mach number.

shown in figure 1, because even in incompressible flow they tend to underestimate the decrease in c_f caused by a suddenly applied pressure gradient. The pressure-gradient parameter $(\delta^*/\tau_w)\,dp/dx$, the ratio of the contributions of pressure gradient and of wall stress to the growth of the momentum deficit in the boundary layer, reaches a maximum of about 4 in this flow. In subsonic flow, the prolonged application of such a pressure gradient would reduce the skin-friction coefficient c_f to perhaps half the constant-pressure value: the calculations for Zwarts' flow follow the trend expected from subsonic flow behaviour, but the experimental values of c_f decrease only slightly in the region of adverse pressure gradient and then rise, significantly exceeding the constant-pressure value indicated by the broken line in figure 1.

Bradshaw & Ferriss (1971) recognized that the hypotheses on which their calculation method was based "would fail in the presence of shock waves and expansion fans where the dilatation div **U** is large", but they did not consider that the dilatation found in distributed pressure gradients could seriously affect the turbulence structure, because div **U** is small to the boundary-layer approximation. They were inclined to attribute Zwarts' results to the effects of three-dimensionality in his rather narrow wind tunnel. Most of the other supersonic boundary-layer measurements available to Bradshaw & Ferriss had been made on highly curved surfaces (used to induce the required pressure gradient) and, because of the uncertainty about the effects of large streamline curvature on turbulence structure, were regarded as unsuitable for testing the calculation method. For a discussion, see §6 of Bradshaw & Ferriss (1971). Apart from Zwarts' flow the only test cases of strong pressure gradient on flat surfaces available to Bradshaw & Ferriss were the accelerated flows of Pasiuk, Hastings & Chatham (1964) and of Sivasegaram (1970): although both showed a smaller c_f than predicted, the evidence was not very conclusive. Since Bradshaw & Ferriss' paper was written, further evidence has accumulated. Bushnell & Alston (1972) concluded that discrepancies between calculation and experiment in various supersonic flows might *not* be entirely attributable to known effects such as those of surface curvature or normal pressure gradient. The experiments of Peake, Romeskie & Brakmann (1972) and of Lewis, Gran & Kubota (1972) on boundary layers rather similar to Zwarts' show similar trends in c_f, and calculations show similar disagreements with experiment: the measurements of Lewis *et al.*, in particular, seem to be of very high quality and free from three-dimensional effects. Fluctuation measurements by Behrens (1971) and Rose (1972) in oblique shock/boundary-layer interactions and by Lewis & Behrens (1969) in the shear layer emerging from a Prandtl–Meyer expansion have shown unexpectedly large increases in turbulence intensity during compression and large decreases during expansion. The only obvious explanation of what now seems to be incontrovertible experimental evidence is that the mean dilatational rate of strain div $\mathbf{U}$ directly affects the turbulence structure to an extent much greater than expected from the terms (in, say, the Reynolds-stress transport equations) that contain the extra rate of strain explicitly.

The main *a priori* reason for believing that compression or dilatation may have large effects on the structure of turbulent shear layers is that other extra rates of strain have been found to change the Reynolds stresses by an order of magnitude more than expected from the size of the explicit extra terms in the Reynolds-stress transport equations. In view of the shortage of turbulence measurements in supersonic flow, many of our conclusions about the effects of compression or dilatation must be inferred from our knowledge of the effects of other extra strain rates in low-speed flows, which are therefore briefly discussed in § 2 in the general context of 'complex' turbulent flows.

In § 3 of the paper we discuss the exact equations for the transport of Reynolds stress in compressible flow, with particular reference to the turbulent-energy equation used by Bradshaw & Ferriss (1971), Wilcox & Alber (1972) and others. We show that an empirical constant in the method of Wilcox & Alber, which they adjust to optimize agreement with experiments on turbulent shear layers passing through shock waves and expansions, is in fact impossibly large for the term it is supposed to represent and should be reinterpreted as an allowance for dilatation effects.

In § 4 we apply a dilatation correction formula, similar to those successfully used to allow for curvature or divergence effects, to the calculation method of Bradshaw & Ferriss (1971). Significant improvement in predictions is found, but because of the abrupt application or release of pressure gradient in the supersonic boundary layers mentioned above it is necessary to make a further simple allowance for the history of the extra rate of strain, derived from physical arguments and applicable to any type of extra strain rate. The final calculations show greatly improved agreement with experiment, as good as can be expected since the extra strain rates are in some places too large for the rigorous application of first-order correction formulae. The modifications to the computer program are minor, and similar allowances for dilatation effects could be incorporated in other field calculation methods, including eddy-viscosity methods: Green, Weeks & Brooman (1972) have used the above-mentioned correction formula, without the allowance for rate-of-strain history, in an integral version of the present calculation method. We may note that although the history allowance is in effect a differential equation for the eddy length scale, a conventional length-scale equation (Bradshaw 1972*b*) will not simulate the effects of extra rates of strain; a correction formula is still needed.

In § 5 we briefly discuss the physics of dilatation effects, which are largely hypothetical because of the lack of turbulence measurements in high-speed flow. The analogy with lateral convergence or divergence, due to Dr J. E. Green (private communication), is at least qualitatively helpful.

The purpose of the present paper is to convince research workers of the reality of the phenomenon, and to encourage engineers to use the method of predicting it described herein. It appears to be essential to make some kind of allowance for the effects of mean dilatation or compression on the turbulence structure in any calculation method for supersonic shear layers (the effects are negligible at Mach numbers significantly less than unity). The phenomenon may be important in turbulent flows other than shear layers: in turbulent combustion the 'mean' dilatation can be of the same order as the fluctuating rate of strain in the larger eddies.

2. 'Complex' turbulent flows

It is helpful to divide turbulent flows with significant Reynolds-stress gradients into (i) simple shear layers and (ii) complex flows.

We define a simple shear layer as one which has a monotonic velocity (strictly, with a shear stress of one sign everywhere) and in which extra strain rates are so small compared with the velocity gradient ($\partial U/dy$ in the usual notation) that they do not significantly affect the turbulence. Because the effect of extra strain rates is or can be an order of magnitude larger than expected from their explicit appearances in the equations, the extra strain rates permissible in a simple shear layer are generally an order of magnitude smaller than those permitted by the thin-shear-layer (boundary-layer) approximation, which is a condition on the relative size of explicit terms.

Complex turbulent flows include *interacting* shear layers, such as occur in ducts or jets, and shear layers perturbed by *extra rates of strain*: a composite type is a shear layer undergoing a change of species (e.g. from a free mixing layer to a boundary layer or vice versa). As a rule, calculation methods for simple shear layers cannot satisfactorily predict the effects of asymmetrical interactions or extra strain rates on turbulence structure without extra hypotheses or empirical information: for general reviews see Bradshaw (1972*a*, 1973*a*,*b*). Here we are concerned with extra strain rates: it is convenient to define *small extra strain rates* as those permitted by the thin-shear-layer approximation, so that each extra strain rate e satisfies

$$e \ll \partial U/dy,$$

where representative rather than local values are understood, and where we consider only two-dimensional flows, for simplicity. *Large extra strain rates* violate the thin-shear-layer approximation and the above inequality; however it appears (Castro 1973; Bradshaw 1973*a*, *b*) that nearly all flows in which Reynolds-stress gradients are significant obey what may be called the *fairly-thin-shear-layer* approximation. To this approximation the effect of Reynolds stresses on the variation of total pressure along a streamline is due principally to the transverse shear-stress gradient referred to axes along and normal to the streamline; normal-stress gradients are small enough for liberties to be taken in calculating them, although they may not be negligible as they are in thin shear layers. This is a useful simplification in the development of calculation methods. The real distinction between small and large strain rates is therefore not the applicability or otherwise of the thin-shear-layer approximation to the Reynolds-stress gradients but the validity or otherwise of first-order empirical correlations of the effects of extra strain rates on the turbulence structure. This point is particularly relevant in supersonic flows, in which shocks and expansions can produce large local extra strain rates in predominantly thin shear layers.

The simplest parameter by which to correlate the effects of an extra strain rate e is the rate-of-strain ratio $e/(\partial U/\partial y)$ or, a rough equivalent, the ratio of e to a typical fluctuating strain rate of the larger eddies, say $eL/(-\overline{uv})^{\frac{1}{2}}$, where L is an eddy length scale such as the dissipation length parameter. The extra strain rate explicitly changes the sum of the 'generation' terms in an exact Reynolds-stress transport equation (e.g. the production term in the turbulent energy equation) by a factor which may be written generally as

$$f = 1 + a\frac{e}{\partial U/\partial y}, \tag{1}$$

where $|a|$ is of order unity (being a ratio of Reynolds stresses) and depends on the form of e. It is found that the implicit effect of almost any extra strain rate on the 'destruction' terms (e.g. the turbulent energy dissipation) is larger than this by an order of magnitude: we call it an implicit effect because the destruction terms, unlike the generation terms, do not contain e explicitly. This suggests that the effect of extra strain rates should be represented in transport-equation calculation methods by dividing (for convenience) the empirical form of the 'destruction' terms by a factor

$$F = 1 + \alpha\frac{e}{\partial U/\partial y}, \tag{2}$$

where $|\alpha|$ is an order of magnitude greater than unity and depends on the form of e.

It should be noted that the effect of extra strain rates on the turbulence structure of a shear layer is of the opposite sign to, as well as an order greater than, the effect of a strain rate on initially isotropic turbulence predicted by rapid-distortion theory. According to Crow's (1968) version of the theory, the pressure-strain term in the Reynolds-stress transport equation acts to oppose the explicit generation terms, but the effects described above imply that the pressure-strain term usually adds overwhelmingly to the explicit terms. Evidently the effects of extra strain rate depend critically on the structure of the turbulence set up by the primary rate of strain $\partial U/\partial y$.

Since α is large the f-factor (1) is much closer to unity than the F-factor (2) and, to the local-equilibrium approximation, we can either divide the destruction terms by F or multiply the generation terms by F. The effects of extra strain rates on turbulent transport terms are not well understood at present, so it is not possible to advance beyond a local-equilibrium discussion. All the above remarks apply equally to transport equations for turbulent energy dissipation or other quantities implying a length scale. In a thin shear layer, where by definition $e \ll \partial U/\partial y$ (small extra strain rates), there is usually no point in making F a non-linear function. Various alternative plausibility arguments can be constructed for (2): for instance, in the case of streamline curvature ($e = \partial V/\partial x$), the rate-of-strain ratio in (2) is to first order equal to half the 'Richardson number' (analogous to the well-known buoyancy parameter: see Bradshaw 1973*a*), and (2) is analogous to the Monin–Obukhov factor. In a calculation method based on the mixing-length formula, the rough correspondence between the mixing length and the dissipation-length parameter used in empirical modelling of the energy dissipation term suggests that the mixing length should be *multiplied* by a factor F; the eddy viscosity should be multiplied by a factor close to F^2.

Correction factors like (2), with $|\alpha| \approx 10$, have been used in several methods for calculating flows with streamline curvature, lateral convergence or divergence.† In convergence or divergence in the plane of the mean shear ($e = \partial V/\partial y$), α seems to be rather smaller: the function of production/dissipation used by Rodi (1972) as a correction factor in his calculation method for free shear layers can be related to (2) with $e = \partial V/\partial y$, because the large growth rate of jets and mixing layers implies both a significant value of $\partial V/\partial y$ and an excess of production over dissipation. Nonlinear versions of (2) have been used for larger extra strain rates (e.g. So & Mellor 1972) but it appears, partly as a result of the present investigation of suddenly applied compression and dilatation rates, that a more necessary improvement is an allowance for the history of the extra strain rate. Such an allowance would be provided automatically by a length-scale transport equation, but in the present work we have used an empirical first-order ordinary differential equation for α, with a 'time constant' (relaxation length) of 10δ in conformity with traditional estimates (Townsend 1956, p. 189) of the response time of the energy-containing eddies. It is convenient to write the equation in terms of the effective value of αe, E say: then

$$10\delta\, dE/dx = \alpha_0 e - E, \tag{3}$$

where α_0 is the asymptotic value of α, reached after prolonged application of the extra strain rate. Note that this ignores the small extra generation terms, represented by (1), which vary directly with the local strain rate. The empirical linear

† It is not known whether skew ($e = \partial W/\partial x$) has large effects; in this case, symmetry would require F to be an *even* function of e.

equation (3) is no more a law of nature than the empirical linear 'F-factor' (2), but both are simple representations of undoubted physical effects and have been used successfully in predictions for suddenly applied curvature (Bradshaw 1973*a*), lateral divergence (Young, unpublished work at Imperial College) and dilatation (see below).

We proceed to discuss the special case of compression or dilatation in a thin shear layer. We shall see that the *a priori* expectation that this extra strain rate, like others, produces large effects on turbulence structure is indirectly confirmed by the success of a correction formula based on (2) and (3).

3. Analysis

The exact equations for the rate of change of Reynolds stresses with time, following the mean motion of the fluid, all contain terms expressing the generation of Reynolds stress by interaction of the turbulence and the mean rate of strain, destruction of Reynolds stress by pressure fluctuations or viscous action, and spatial transport of Reynolds stress by velocity fluctuations, pressure fluctuations or viscous action. Although our main interest is in the shear stress, the turbulent energy equation for the sum of the normal stresses behaves similarly, is easier to discuss and better documented experimentally, and is used in the calculation methods of Bradshaw & Ferriss, Wilcox & Alber and others.

Bradshaw & Ferriss (1971), using tensor suffix notation and conventional time averages of velocity so that the mean value of the fluctuating velocity $\overline{u}_i$ is zero, give the equation for the turbulent energy per unit volume as

$$\underset{\text{(i)}}{U_j\partial(\tfrac{1}{2}\overline{\rho}\,\overline{u_i^2}+\tfrac{1}{2}\overline{\rho' u_i^2})/\partial x_j} = \underset{\text{(ii)}}{-(\overline{\rho}\,\overline{u_i u_j}+\overline{\rho' u_i u_j})\,\partial U_i/\partial x_j}$$
$$\underset{\text{(iii)}}{-\,\partial(\overline{p' u_j}+\tfrac{1}{2}\overline{\rho}\,\overline{u_i^2 u_j}+\tfrac{1}{2}\overline{\rho' u_i^2 u_j})/\partial x_j} - \underset{\text{(iv)}}{\text{viscous diffusion and dissipation}}$$
$$\underset{\text{(v)}}{-\overline{\rho' u_i}\,U_j\partial U_i/\partial x_j} - \underset{\text{(vi)}}{(\tfrac{1}{2}\overline{\rho}\,\overline{u_i^2}+\tfrac{1}{2}\overline{\rho' u_i^2})\,\partial U_j/\partial x_j} + \underset{\text{(vii)}}{\overline{p'\,\partial u_j/\partial x_j}}. \qquad (4)$$

In incompressible flow terms (v)–(vii) disappear, and the elements of the energy production term (ii) for which $i=j$ almost cancel because $\operatorname{div}\mathbf{U} \equiv \partial U_j/\partial x_j = 0$. Terms (i) and (vi) combine to give

$$(\overline{\rho}U_j+\overline{\rho' u_j})\frac{\partial}{\partial x_j}\left(\frac{1}{2}\left\{\overline{u_i^2}+\frac{\overline{\rho' u_i^2}}{\overline{\rho}}\right\}\right) - \frac{\partial}{\partial x_j}\left(\overline{\rho' u_j}\frac{1}{2}\left\{\overline{u_i^2}+\frac{\overline{\rho' u_i^2}}{\overline{\rho}}\right\}\right),$$

so that the mean dilatation $\partial U_j/\partial x_j$ does not appear explicitly in the equation for the turbulent energy per unit *mass*, $\frac{1}{2}\overline{u_i^2}$. Bradshaw & Ferriss showed that (vii) should be small compared with the pressure term in (iii) and therefore negligible in most practical cases. They retained part of (v), the turbulent mass flux times the mean acceleration, neglecting the contribution of the pressure gradient to the mean acceleration: they also neglected the normal-stress elements of (ii), which sum to rather less than $-\frac{1}{3}\overline{\rho}\overline{u_i^2}\,\partial U_j/\partial x_j$ and should therefore be small in a weak pressure gradient. Bradshaw & Ferriss assumed that these small terms in the turbulent-energy equation would have little influence on a flow dominated by strong pressure gradients: unfortunately this is true only in the letter and not in the spirit because these are the terms which contain the dilatation. The neglected part of (v), $(\overline{\rho' u_i/\bar{p}})\,\partial\bar{p}/\partial x_i$, closely equal to $(\overline{\rho' u/\bar{p}})\,d\bar{p}/dx$ in a thin shear layer, can be estimated by using Morkovin's 'strong Reynolds analogy', which implies that the instantaneous total temperature is constant and the pressure fluctuations negligibly small. This leads to

$$\rho'/\bar{\rho} \approx (\gamma-1)\,M^2 u/U, \tag{5}$$

so that $\overline{\rho' u} \approx (\gamma-1)\,M^2 \overline{\rho u^2}/U$. In the free stream, the continuity equation and the gas laws give

$$\operatorname{div}\mathbf{U} = -(U/\gamma\bar{p})\,d\bar{p}/dx, \tag{6}$$

which should be an adequate approximation within the shear layer because errors will be appreciable only near the surface, where $\operatorname{div}\mathbf{U}/(\partial U/\partial y)$ is small. Taking $\overline{u_i^2} \equiv \overline{q^2} \approx 2{\cdot}5\overline{u^2}$, we find that the sum, T_e say, of the normal-stress part of (ii) and the pressure-gradient part of (v) for a gas with $\gamma = 1{\cdot}4$ is

$$T_e \approx -0{\cdot}5\bar{\rho}\,\overline{q^2}\operatorname{div}\mathbf{U} \approx 0{\cdot}5\overline{q^2}\,\frac{U}{a^2}\frac{d\bar{p}}{dx}, \tag{7}$$

where a is the speed of sound. The result should be accurate to about one significant figure. Now the main production term in a thin shear layer is (ii) with $i = 1$ and $j = 2$, or $-\bar{\rho}\overline{uv}\,\partial U/\partial y$ if we use ordinary notation and neglect the density fluctuations. Using Bradshaw & Ferriss' approximation $-\overline{uv} \approx 0{\cdot}15\overline{q^2}$ we see that the ratio of the dilatation terms to the main production terms is about $3\operatorname{div}\mathbf{U}/(\partial U/\partial y)$.

Wilcox & Alber write the turbulent-energy equation in terms of mass-weighted averages and the pressure-gradient part of (v) appears explicitly. They follow the above estimate of the dilatation terms T_e, but replace the factor 0·5 with an empirical constant $\frac{1}{2}\zeta$: to secure agreement with experiment they put $\zeta = 2{\cdot}5$–$2{\cdot}7$; that is, the effects of dilatation are simulated by increasing the explicit dilatation terms T_e by a factor of about 2·6. Wilcox & Alber do not mention that this is an unphysically large value for T_e. The effect of Wilcox & Alber's procedure is the same as neglecting the dilatation terms and multiplying the main production term by

$$F = 1 - 8\,\frac{\operatorname{div}\mathbf{U}}{\partial U/\partial y}, \tag{8}$$

a special case of (2). No great significance is attached to the numerical constant because Wilcox & Alber use the turbulent energy only as part of an eddy viscosity, but it is safe to deduce that the effects of dilatation and compression on the turbulence structure of the shock and expansion flows used to calibrate Wilcox & Alber's method was considerable. For the sake of retaining a positive value of α, as in the cases $e = \partial V/\partial x$ and $e = \partial W/\partial z$, we shall regard a compressive strain rate as positive: $e = -\operatorname{div}\mathbf{U}$.

In the free stream of a two-dimensional flow,

$$\operatorname{div}\mathbf{U} \equiv \partial U/\partial x + \partial V/\partial y = M^2\,\partial U/\partial x \approx \partial V/\partial y$$

if M is large compared with unity. As mentioned in §2 the effects of $\partial V/\partial y$ as such seem to be rather smaller than those of other extra strain rates: negative $\partial V/\partial y$, alone, appears to reduce the shear stress, while positive $\partial W/\partial z$, associated with negative $\partial V/\partial y$ by the continuity equation, increases the shear stress. The effects of $\partial V/\partial y$, as such, and those of div **U** will coexist, but it is simpler to correlate entirely in terms of div **U** in compressible flow.

4. Modifications to the method of Bradshaw & Ferriss

Wilcox & Alber did not publish comparisons of their method with any of the recent measurements of boundary layers in distributed pressure gradients, so that as a further test of the applicability of (1) we made a modification, similar to that described by (8), to the method of Bradshaw & Ferriss. This method uses the assumption of a one-to-one correspondence between the shear-stress profile and the profile of any other turbulence quantity to convert the turbulent energy equation into an empirical transport equation for the shear stress. The process is broadly similar to Wilcox & Alber's modelling of the turbulent-energy equation except that the shear stress is related to the turbulent energy directly and not via an eddy viscosity. Wilcox & Alber solve a second transport equation to provide an eddy length scale but Bradshaw & Ferriss assume that the dissipation length scale L appearing in their empirical equation for the shear stress is given by $L/\delta = f(y/\delta)$, where f is an empirical function. In the local-equilibrium approximation (Townsend 1961) L becomes equal to the apparent mixing length. Bradshaw & Ferriss' final equation for $-\overline{uv}$, with the true dilatation terms obtained from (7), is

$$\frac{D(-\overline{uv}/(2a_1))}{Dt} = (-\overline{uv})\frac{\partial U}{\partial y} - \frac{(-\overline{uv})^{\frac{3}{2}}}{L} - \frac{0\cdot 5}{a_1}(-\overline{uv})\,\mathrm{div}\,\mathbf{U} + \ldots, \tag{9}$$

where $a_1 \equiv -\overline{uv}/\overline{q^2} \approx 0\cdot 15$ and where the main production term is the first term on the right. The correction for dilatation consists of dividing the dissipation term by F from (2). To the local-equilibrium approximation and the first order in rate-of-strain rates, this is equivalent to subtracting $\alpha(-\overline{uv})\,\mathrm{div}\,\mathbf{U}$ from the right-hand side of (9) and, for simplicity, we adopt this approximation and also absorb $0\cdot 5/a_1$ into α. In the present computer program we have approximated div **U** by $-(U/\gamma\overline{p})\,d\overline{p}/dx$.

Figure 2 shows calculations† for the retarded/accelerated flow of Lewis *et al.*, for $\alpha = 0, 5, 7$ and 10. Note that in this case the surface shear stress follows the *qualitative* trend familiar from low-speed flows, with a decrease in the adverse pressure gradient and an increase in the favourable pressure gradient. However, the unmodified calculations ($\alpha = 0$ in figure 2) lie well below the data, as in Zwart's flow (figure 1). Figure 2 shows rough values of the pressure-gradient parameter $(\delta^*/\tau_w)\,dp/dx$ and of div $\mathbf{U}/(\partial U/\partial y)$, taking $\partial U/\partial y = 0\cdot 3U_e/\delta$ as a typical value in the middle of the boundary layer: to this approximation

$$\mathrm{div}\,\mathbf{U}/(\partial \mathbf{U}/\partial y) = 3M_e^2(U_e/\delta)\,dU_e/dx,$$

† The Reynolds number of the experiment was rather low: the Reynolds-number correction factor used in the program was adjusted to give agreement with the constant-pressure results of Lewis *et al.*

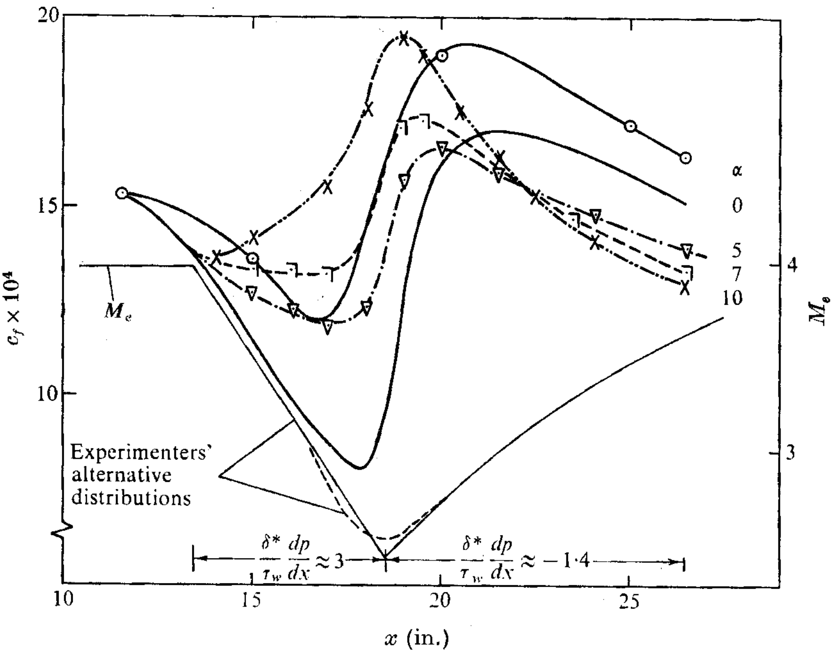

FIGURE 2. Experiment of Lewis *et al.* —⊙—, mean line through data; ——, calculations without allowance for dilatation. Calculations with algebraic equation (2) for dilatation: ▽, $\alpha = 5$; ⌝, $\alpha = 7$; ×, $\alpha = 10$. Div $\mathbf{U}/(\partial U/\partial y)$ reaches about $-0{\cdot}1$ in the retarded region, $0{\cdot}14$ in the region of acceleration.

showing that the effects of dilatation vary roughly as M_e^2. The general trend of the calculations is improved by the empirical correction for dilatation, but a value of α large enough to give the right maximum c_f predicts a rise in c_f right from the start of the adverse pressure gradient, and c_f decreases again as soon as the pressure gradient becomes favourable. Evidently the full effect of dilatation on the turbulence structure is not felt as soon as the dilatation begins: presumably the initial effect is that of the real increase in the generation terms, while the effect of the empirically added term, representing a change in the turbulence structure, increases gradually. Similar effects have been found on surfaces with sudden changes of curvature by Thomann (1968) and by Young in unpublished work at Imperial College.

Further calculations, including a comparison with the measurements of Winter, Rotta & Smith (1968) on a waisted body of revolution, have been done by Green *et al.* (1972). Again, significant improvements were found when an allowance for dilatation, based on (2) with $\alpha = 7$, was made.

Figure 3 shows the effect of using the 'lag equation' (3) for the effective value of $-\alpha \operatorname{div} \mathbf{U}$ (actually for the effective value of $\alpha(1/\gamma\bar{p})\, d\bar{p}/dx$, which is independent of y). α_0 was taken as 10. There is clearly a further improvement, although c_f (figure 3*a*) still falls more rapidly than it should in the region of favourable pressure gradient, suggesting that α should be smaller, or the relaxation distance larger than 10δ, when div $\mathbf{U}$ is positive than when it is negative. Similar effects,

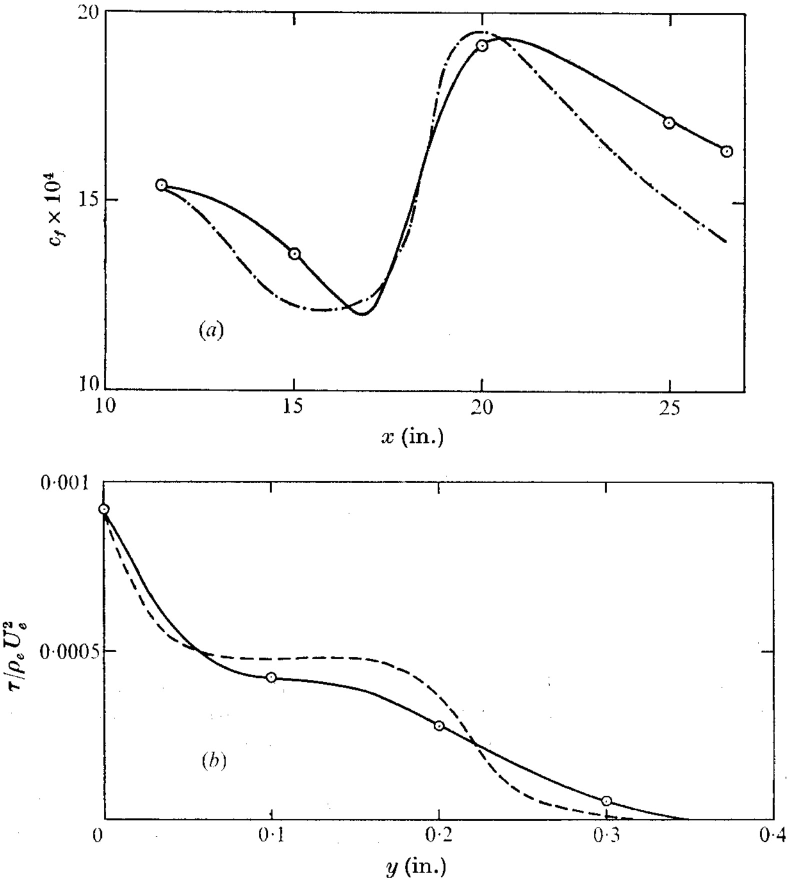

FIGURE 3. Experiment of Lewis *et al.* (*a*) Surface shear-stress. —⊙—, experiment; – · – · – ·, calculation with rate equation (3) for effective value of αe in (2), $\alpha_0 = 10$. (*b*) Shear-stress profile at $x = 21{\cdot}5$ in. —⊙—, experiment (from total-pressure profiles); – – – –, calculation as in (*a*).

in the opposite sense, are found in curved flows, where the best-straight-line approximations imply that $\alpha \approx 14$ in stable situations and $\alpha \approx 8$ in unstable situations. However, some of the assumptions made in the calculation method of Bradshaw & Ferriss (1971) are valid only if $(\gamma - 1)M^2$ is not much greater than unity, and this condition is poorly satisfied in the outer layer of a boundary layer with $M_e \approx 4$. Moreover the values of div $\mathbf{U}/(\partial U/\partial y)$ in the flow of Lewis *et al.* are rather too large for the confident application of any first-order formula throughout the layer: however, it is well known that the outer region of a boundary layer in a strong pressure gradient is dominated by pressure gradients rather than Reynolds-stress gradients; in the inner layer, where Reynolds-stress gradients are more important, div $\mathbf{U}/(\partial U/\partial y)$ is smaller because $\partial U/\partial y$ is larger. The present formula does appear to give tolerably good results in the outer layer even in the flow of Lewis *et al.*, partly because the full effects of an extra strain rate are felt only if it is applied for a downstream distance of many boundary-layer thicknesses, a phenomenon here crudely represented by (3).

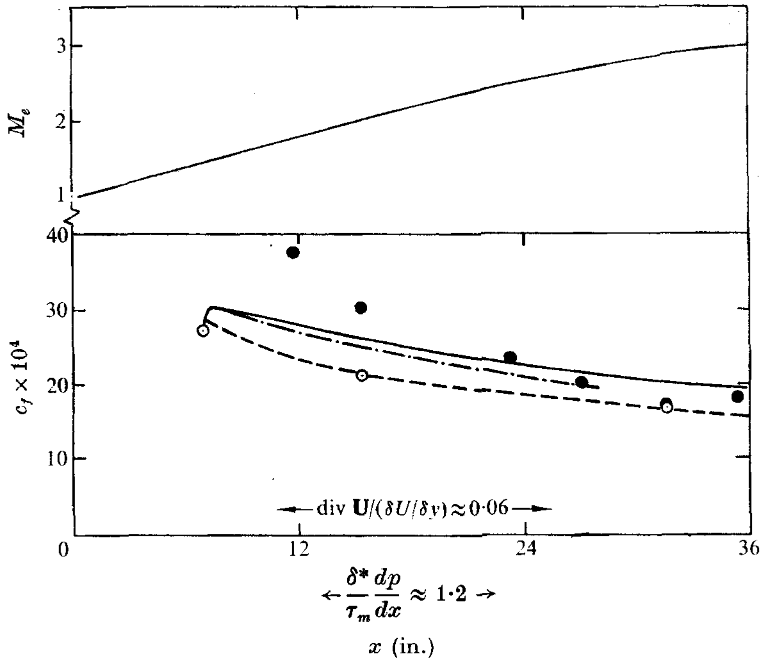

FIGURE 4. Experiment of Pasiuk *et al.* --⊙--, experiment (from logarithmic profiles); ●, experiment (from momentum integral); ——, calculation without allowance for dilatation; —·—·, calculation with allowance for dilatation, (2) with $\alpha = 7$.

Figure 3(*b*) shows the calculated shear-stress profile at $x = 21{\cdot}5$ in. together with that obtained from the experimental total-pressure profiles via the useful but apparently little-known expression

$$\frac{\partial \tau}{\partial y} = \frac{p}{P}\left(\frac{\partial P}{\partial x}\right)_{\text{streamline}}, \tag{10}$$

which is exact if the total temperature is constant. It was not possible to measure $\partial P/\partial x$ with any accuracy in the region of rapid change between $x = 17$ in. and 19 in. but the shear stress in mid-layer appears to be at least five times larger at $x = 17$ in. than at $x = 21{\cdot}5$ in.: the calculated values of $\tau_{\max}/\rho_e U_e^2$ were 0·0021 at $x = 17$ in. and 0·0018 at $x = 19$ in., both at about $y/\delta = 0{\cdot}5$. The fall to less than 0·0005 at $y/\delta = 0{\cdot}5$ by $x = 21{\cdot}5$ in. represents a fourfold decrease in a streamwise distance of 10δ.

Calculations for the experiments of Pasiuk *et al.*, Peake *et al.*, Zwarts and Waltrup & Schetz (1972) are shown in figures 4–7. The Pasiuk calculation was done early in the investigation, with $\alpha_0 = 7$: a *much* larger value would be needed to achieve agreement with values of c_f deduced from the logarithmic part of the velocity profiles. The evidence from this experiment and from the accelerated part of the flow of Lewis *et al.* is therefore contradictory: in the absence of other evidence we must follow Wilcox & Alber's lead and use the same empirical constants for both signs of div $\mathbf{U}$. The results for the retarded flows of Peake *et al.* and of Zwarts, both with $\alpha = 10$, seem satisfactory, the remaining discrepancies being in opposite directions in the two experiments. The results for the retarded

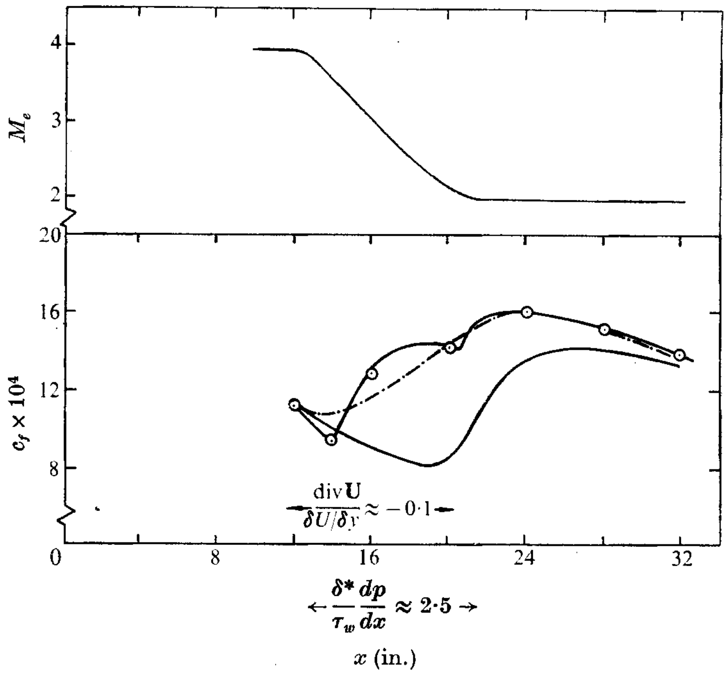

FIGURE 5. Experiment of Peake *et al.* —⊙—, mean line through data; ——, calculation without allowance for dilatation; —·—·—·—; calculation with rate equation (3) for effective value of αe in (2), $\alpha_0 = 10$.

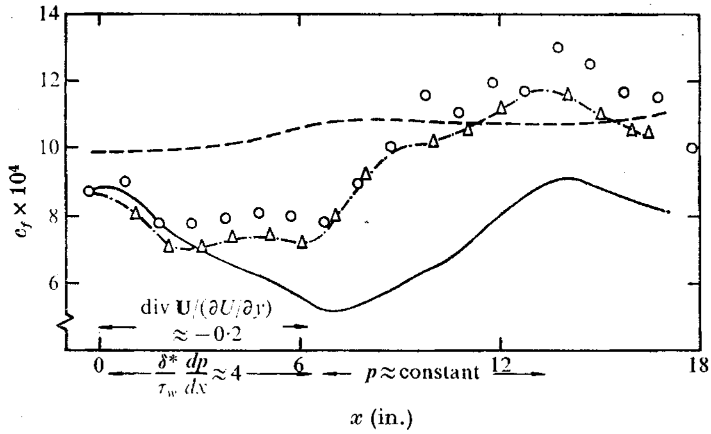

FIGURE 6. Experiment of Zwarts. —·—·—, calculation with allowance for dilatation, (2) and (3) with $\alpha_0 = 10$. For rest of legend see figure 1.

flow of Waltrup & Schetz are less satisfactory but may at least be taken as further qualitative evidence for dilatation effects.

The correction for compression and dilatation effects represented by (2) and (3), with $\alpha_0 = 10$ as the asymptotic value of the empirical constant α in (2), is offered as a simple and convenient package for insertion into other calculation methods. It is nominally limited to cases where div **U** is small compared with $\partial U/\partial y$ or $(-\overline{uv})^{\frac{1}{2}}/L$, because there is no reason to expect (2) to remain linear for large strain

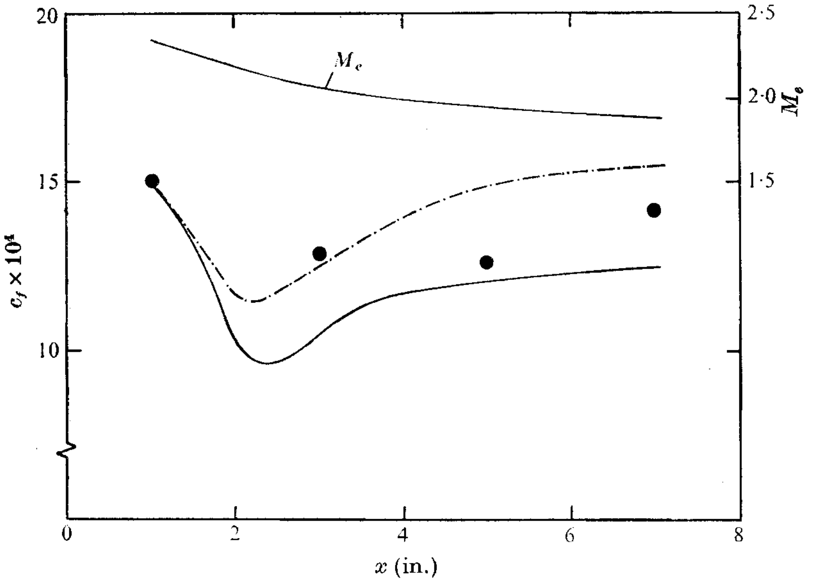

FIGURE 7. Experiment of Waltrup & Schetz. ●, experiment (ramp 1); ——, calculation without allowance for dilatation; —·—·—, calculation with rate equation (3) for effective value of αe in (2), $\alpha_0 = 10$.

rates. Also, the allowance for rate-of-strain history (3) is clearly a crude approximation to what should strictly be a partial differential equation. However, any improvements on (2) and (3) may have to be too closely integrated with a given calculation method to be easily applicable in general. The current version of Bradshaw & Ferriss' calculation method contains an extension of (2) and (3) to cope with several types of extra strain rate at once: (2) and (3) appear to give tolerably good results even in flows with short regions of large extra strain rate ('strain impulses') for reasons explained earlier in this section.

5. Physics of dilatation effects

There is no direct evidence for the mechanism by which dilatation reduces, and compression increases, the turbulence intensities. Dr J. E. Green has pointed out to me that compression in the x, y plane increases the z component of vorticity in the same way as lateral divergence: both influences cause a reduction in cross-sectional area of a fluid element in the x, y plane and thus, if angular momentum is conserved, an increase in vorticity. Div $\mathbf{U}$ is equivalent to $-\partial W/\partial z$. There are bound to be detailed differences in the mechanisms, but a qualitative correspondence seems almost certain and lends credence to the empirical arguments of § 3. Unfortunately the effects of lateral divergence are themselves not well understood: lateral divergence increases the mean vorticity as well as the fluctuating vorticity, so it is not immediately obvious why the Reynolds stresses should differ greatly from those predicted by, say, a conventional eddy viscosity. Keffer (1965, 1967) has found that the large eddies in a wake, which have a strong z component of vorticity in any case, are amplified by divergence while the smaller eddies, being more nearly isotropic, are if anything suppressed. It seems likely that the same effects occur in a boundary layer. Keffer found that lateral convergence apparently had little effect but the measurements of Patel, Nakayama & Damian (1974) near the tail of a body of revolution suggest that convergence produces effects of the same order as divergence, but of the opposite sign. Nothing can be inferred from this about the equality or otherwise of the optimum values of α for compression and dilatation, and further speculation in the absence of evidence is unlikely to be helpful.

6. Conclusions

Comparisons with recent, well-conducted measurements on supersonic boundary layers in moderate or strong pressure gradients have revealed large inaccuracies in conventional calculation methods. The evidence suggests that bulk compression or dilatation, div **U**, has a much greater effect on the turbulence structure of a shear layer than is expected from the size of the extra terms, in the Reynolds-stress transport equations or other turbulence equations, which explicitly contain div **U**. This behaviour has previously been found in the case of other 'extra strain rates' such as lateral divergence or longitudinal curvature of the streamlines, and an analogy can by drawn between the effects of compression and of lateral divergence on the cross-sectional area, in the plane of the mean shear, of a fluid element or vortex line. An empirical correction formula of a type found satisfactory in predicting the effects of divergence or curvature significantly improves agreement between the calculation method of Bradshaw & Ferriss (1971) and experiment. A new feature, found in the present work on suddenly applied pressure gradients and in unpublished work at Imperial College on suddenly applied curvature and divergence, is that the correction formula is much improved by an allowance for rate-of-strain history, in addition to any such allowance in the basic calculation method.

Although the picture of the phenomenon offered in this paper has been pieced together from indirect arguments, analogies and empirical modifications of an empirical calculation method, there seems little doubt that an unexpected effect of dilatation on shear-layer turbulence exists, and that it must be allowed for in engineering calculation methods for supersonic shear layers. The correction formula suggested here could be applied to most types of calculation method.

From the fundamental viewpoint, the addition of another member to the family of unexpectedly powerful extra strain rates enhances the importance of 'complex' turbulent flows as a research topic, and further degrades the status of the simple shear layer as a sufficient source of information about turbulence structure. We cannot expect extensive data on turbulence structure in supersonic flow to appear in the near future, but this increases the need for experiments on more tractable examples of extra strain rates.

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